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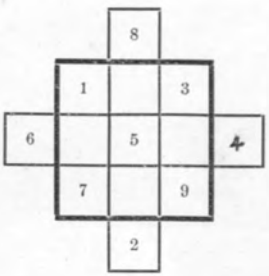
ANALYSIS

OF THE

PROPERTIES OF NUMBERS

AS DISCOVERED AND APPLIED TO THE
CONSTRUCTION OF

MAGIC SQUARES.



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BY

BENJAMIN F. MARSH, C. E.

TOGETHER WITH METHOD OF FINDING TRUE MERIDIAN BY OBSERVATION ON POLARIS.

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PREFACE.

THIS pamphlet treats of special laws of numbers by which problems that have engaged the attention of eminent mathematicians are solved with facility. The principles and methods are scientific, and their absence from all scientific works or Encyclopedias is a singular fact.

BENJAMIN F. MARSH.

HELENA, MONTANA, May 10, 1892.

MAGIC SQUARES.

ARTICLE 1.—DEFINITIONS. A magic square is one divided into a number of small equal squares having an equal number of small squares on each side and a series of numbers each of which is greater or less than the preceding one by a constant difference, so placed in the small squares that the sum of each row, column and diagonal, is a constant number.

The small squares are called cells; when vertical they are called columns, when horizontal they are called rows, the numbers of the series are called terms, of which the first and last are called extremes.

FIG. 1.

4	9	2
3	5	7
8	1	6

MAGIC SQUARE.

The following consequences are deduced:

2. The sum of the series is divided into equal parts and their number is the same as the number of cells in one side of the square.

3. The constant number and the number of cells of one side of the square are factors whose product is the sum of the series.

To find the sum of the series (see progressions in arithmetic) multiply the half sum of the extremes by the number of terms.

Let it be required to demonstrate the following:

THEOREM.

If the cells of a square be filled by the terms of a series in consecutive order, then special properties of numbers exist, from which rules for the construction of magic squares are deduced.

ODD NUMBERS.

4. Let a series of natural numbers having an odd number of terms be set in regular order, reading each row from left to right, then the odd terms are ranked in diagonal rows alternate with parallel rows of the even terms, and the term in the central cell is the half sum of the extremes.

FIG. 2.

1	2	3
4	5	6
7	8	9

FIG. 3.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

5. By Article 3 we find the constant number of the squares of 3, 5 and 7 to be respectively 15, 65 and 175. But the sum of the diagonals of the square are each respectively 15, 65 and 175; hence we discover a special law of numbers; *namely:*

FIG. 4.

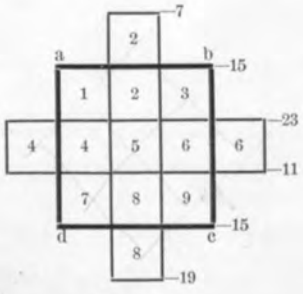
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49

6. ***The sum of the terms of each diagonal is the constant number of the magic square.***

SQUARE OF 3.

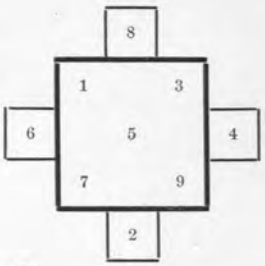
7. Construct the square a, b, c, d, add a cell to the middle one outside the square, set the terms from 1 to 9, reading each row from left to right, remove each even term to the adjacent outside cell and add the terms of each diagonal row. The sum of each diagonal is 15, the constant number, and 7 plus 23 is twice the constant number. Now 23 minus 7 is equal to 16, and if

FIG. 5.



half this difference, 8, is subtracted from the greater sum, 23, and added to the lesser, 7, then each would be 15. But we find the required difference between the even terms of the same rows, for 8 minus 2 is equal to 6, and 6 minus 4 is equal to 2, and the sum of these differences is 8. To balance sums of rows it is only necessary to exchange the even terms in the order named. Therefore to construct the magic square remove the even terms along the row or column in which they stand to the remote instead of the adjacent outside cell. Collecting the results we construct the magic square, Fig. 6.

FIG. 6.



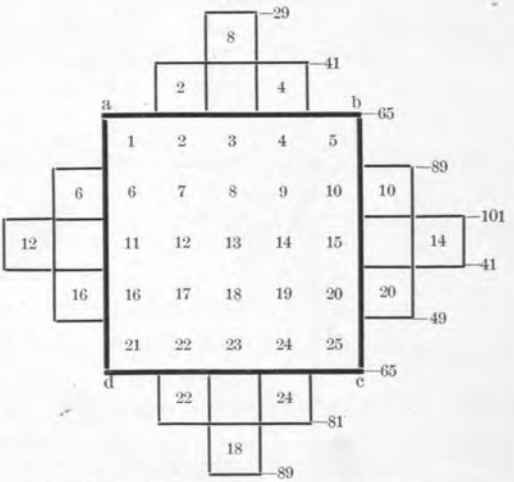
SQUARE OF 5.

8. The series in Fig. 7 is set in the square a, b, c, d, in rows from left to right, even terms removed to adjacent outside DIAGONAL cells and the sum of each diagonal row set at the outside corners. It now appears that the sum of each diagonal of the square is 65, the constant number.

That the sum of each pair of rows parallel to and equidistant from the diagonals is twice the constant number.

And that of the even terms, four groups of three terms each, are in the diagonal cells on each side of the square, a, b, c, d.

FIG. 7.



The difference in these groups of even terms is such that if each term is transposed to the opposite side of the square, a, b, c, d, then ONE-HALF the difference between the members of each pair of rows will be taken from the greater and added to the lesser sum, and the sum of each row will be 65.

For in Fig. 8 this transfer has been made and by addition the constant number, 65, is found. Proof by comparison of differences is obvious by inspection of the diagrams.

9. Figures 7 and 8 show that to construct the magic square, each even term has been moved along the row or column in which it stood to a remote outside cell, a distance of 6 cells, counting one at the place of starting, or *the distance moved is the number of cells of one side of the square plus one.*

FIG. 8.

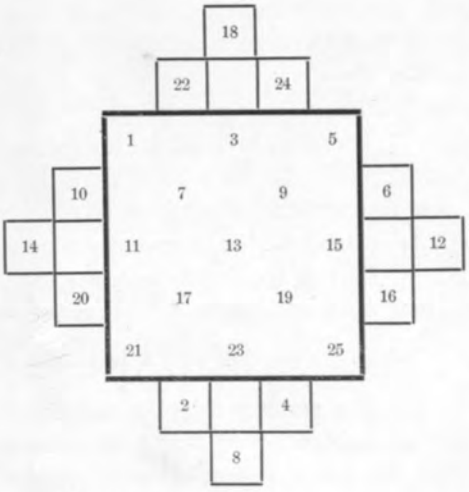
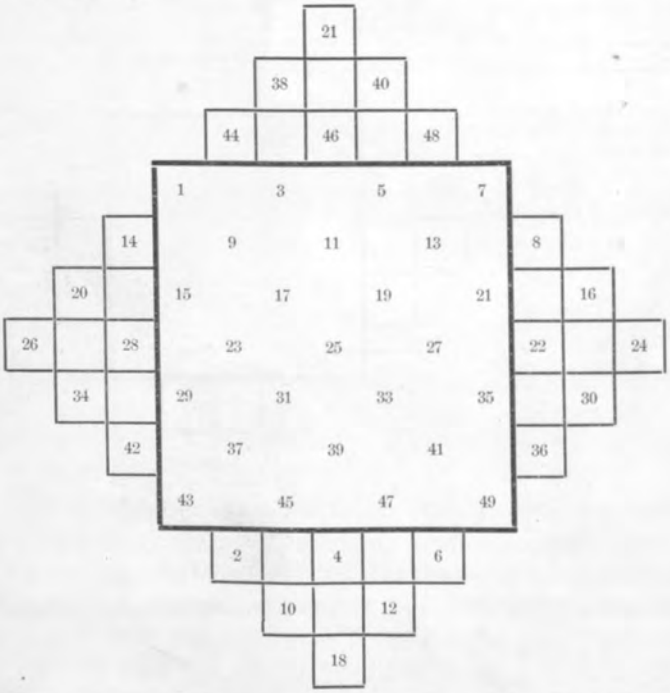


FIG. 9.



This square of 7 is a verification of the rule.

10. General rule for constructing magic squares of odd terms:

Set the terms in rows from left to right, construct a system of diagonal cells by adding them to the outside of the square and remove each even term along the row or column in which it stands a distance counted by the number of cells of one side of the square plus one.

In like manner construct squares of 9, 11, 13, 15, etc., ad infinitum.

11. Magic squares of any odd series of terms may be constructed in like manner.

EXAMPLE 1.

13	3	17
15	11	7
5	19	9

3 TO 19.

EXAMPLE 2.

20	5	26
23	17	11
8	29	14

5 TO 29.

EXAMPLE 3.

12	2	16
14	10	6
4	18	8

2 TO 18.

SQUARES OF EVEN NUMBERS.

SQUARE OF 4.

12. Set the series as shown in Fig. 10. The sum of the columns and diagonals is severally 34, the constant number.

FIG. 10.

1	2	3	4	—10
8	7	6	5	—26
12	11	10	9	—42
13	14	15	16	—58

There are two pairs of rows, the outside and middle rows, and the sum by pairs is 68, or twice the constant number. The half difference between the top and bottom row is 58 minus 10, which divided by 2 is equal to 24. It is obvious by inspection that the difference between the upper

and lower terms of the middle columns is 24, and their exchange will make the sums of rows equal. The difference between the terms of the middle columns in the middle rows is equal to the half difference of sums of middle rows, and their exchange will give equal sums of rows. Hence, to obtain equal sums of rows, invert the middle columns.

By comparing differences of terms of outside columns with the same rows in pairs, we find by their inversion equal summation is the result for the same reason.

Therefore, to convert Fig. 10 to a magic square,

14. RULE.— Invert the middle columns vertically, or,

Invert outside columns.

In Fig. 10 there are obviously two laws of numbers due to the order of filling the square.

FIG. 11.

1	14	15	4
8	11	10	5
12	7	6	9
13	2	3	16

FIG. 12.

13	2	3	16
12	7	6	9
8	11	10	5
1	14	15	4

First. The unequal sums of rows exist in pairs whose aggregate sum is twice the constant number.

Second. The difference between the terms of the middle columns, standing in each pair of rows is equal to the half difference of the sums of each pair of rows, and the reversal of the middle columns takes the half difference from the greater and adds it to the lesser sum of each pair.

The same is true of the outside columns.

14. If a different order of terms is substituted, the law of pairs, aggregating twice the constant number, and the law of special differences in the terms of the columns exists and equal summation is obtained with facility.

15. A multitude of different forms is derived from Figures 11 or 12, all of which are magic squares, because a change in the order of terms to be added makes no change in their sum.

Exchange outside columns Fig. 1, we have Fig. 2.

Exchange top and bottom rows Fig. 2, we have Fig. 3.

Exchange middle columns with outside columns of Fig. 3, we have Fig. 4.

Exchange middle rows with outside rows of Fig. 4, we have Fig. 5.

Exchange middle rows of Fig. 5, we have Fig. 6.

If we continue the process until every term of the series is made to occupy each cell of the square, the number of distinct transformations will be 256.

In like manner squares of any arithmetical series of 16 terms can be made.

FIG. 1.

13	2	3	16
8	11	10	5
12	7	6	9
1	14	15	4

FIG. 2.

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

FIG. 4.

14	4	1	15
11	5	8	10
7	9	12	6
2	16	13	3

FIG. 3.

4	14	15	1
5	11	10	8
9	7	6	12
16	2	3	13

FIG. 5.

11	5	8	10
14	4	1	15
2	16	13	3
7	9	12	6

FIG. 6.

11	5	8	10
2	16	13	3
14	4	1	15
7	9	12	6

SQUARE OF 6.

SERIES OF NATURAL NUMBERS FROM 1 TO 36.

16. Since two of the factors of 36 are 4 and 9, this series contains 9 groups of 4 terms each. These groups form a series of 9 terms, the extremes are 10 and 138 and common difference 16. By Article 7 a magic square can be constructed of this series, but the terms of the original series would be lost. To retain them let the device of the square of 2, containing 4 cells, be adopted. For convenience call them blocks of 4. They will contain the original and new series and are to be employed in the following manner: In their cells set the original series so that the sums of the top and bottom terms shall be equal, thus:

BLOCK 1.

1	4
3	2

It will soon appear that this arrangement secures equal summation of rows.

Number the blocks in the successive order of the new series from 1 to 9.

Draw Diagram No. 13, making the lines so as to indicate clearly the blocks of 4 and the cells in which the original series will appear.

DIAGRAM 13.

21 24 **6** 23 22	1 4 **1** 3 2	29 32 **8** 31 30
25 28 **7** 27 26	17 20 **5** 19 18	9 12 **3** 11 10
5 8 **2** 7 6	33 36 **9** 35 34	13 16 **4** 15 14

Employ Fig. No. 6 for the model by which the new series is to be set so as to be a magic square, and in the center of each block of 4 write the numbers from 1 to 9 in the precise order of the model. These numbers definitely locate each block of 4. They are now to be carried each by number to the assigned place, and Diagram No. 13 is the result of the operation.

Summation of columns and rows shows that the sums of the rows are equal; that the columns are alternately 108 and 114, the sum of 108 and 114 is twice the constant number and their half difference is 3.

DIAGRAM 14.

21 24 23 22	1 4 3 2	29 32 31 30
25 28 27 26	20 17 19 18	9 12 11 10
8 5 7 6	33 36 35 34	16 13 15 14

MAGIC SQUARE.

Now this inequality disappears if in the second block the terms 8 and 5, in the centre block 17 and 20, and in the fourth 13 and 16, be exchanged, for in each case the half difference will be taken from the greater and added to the lesser sum.

DIAGRAM 15.

1 4 3 2		53 56 55 54		57 60 59 58		13 16 15 14	
32 29 30 31		44 41 42 43		40 37 38 39		20 17 18 19	
48 45 46 47		28 25 26 27		24 21 22 23		36 33 34 35	
49 52 51 50		5 8 7 6		9 12 11 10		61 64 63 62	

MAGIC SQUARE.

The properties of numbers before mentioned exist in this case.

17. Since 4 multiplied by 16 is equal to 64 we can in like manner construct the magic square of 8 by the device of blocks of 4, using any model of the square of 4.

18. Since the factors of 81, 9 multiplied by 9, are square numbers, we may form a new series of 9 terms and by

any model of the square of 3 set each block of 9 terms in its assigned place and construct a magic square containing nine magic squares. [Fig. 16.]

DIAGRAM 16.

51	46	53	6	1	8	69	64	71
52	50	48	7	5	3	70	68	66
47	54	49	2	9	4	65	72	67
60	55	62	42	37	44	24	19	26
61	59	57	43	41	39	25	23	21
56	63	58	38	45	40	20	27	22
15	10	17	78	73	80	33	28	35
16	14	12	79	77	75	34	32	30
11	18	13	74	81	76	29	36	31

19. Series from 1 to 100, constant number 505. Two factors of 100 are 4 and 25. We may therefore employ blocks of 4, and the model of the square of 5 (9), and construct the square in the manner described in Article 16. (Fig. 17.)

20. In like manner we may construct the square of 15, employing a model of the square of 7 with blocks of 4.

DIAGRAM 17.

56	53	40	37	4	1	88	85	72	69
54	55	38	39	3	2	86	87	70	71
80	77	41	44	28	25	9	12	96	93
78	79	43	42	27	26	11	10	94	95
81	84	65	68	49	52	33	36	17	20
83	82	67	66	50	51	35	34	19	18
5	8	89	92	73	76	57	60	21	24
6	7	90	91	75	74	59	58	22	23
32	29	16	13	97	100	64	61	48	45
30	31	15	14	99	98	63	62	46	87

MAGIC SQUARE.

21. Transform the multiplication table into a magic square containing 9 magic squares. First make a series of 9 magic squares, from the original series, each containing 16 terms. Each

of the magic squares may be made by a different model (Article 14).

Insert each block of 16 in the order of the numbers in model No. 5. Diagram 18 shows the result from which thousands of combinations can be made.

DIAGRAM 18.

94	84	81	95	1	14	15	4	123	120	117	122
91	85	88	90	8	11	10	5	126	113	116	127
87	89	92	86	12	7	6	9	114	125	128	115
82	96	93	83	13	2	3	16	119	124	121	118
104	107	106	101	78	65	68	79	48	34	35	45
97	110	111	100	75	72	69	74	37	43	42	40
109	98	99	112	71	76	73	70	41	39	38	44
108	103	102	105	66	77	80	67	36	46	47	33
29	18	19	32	130	144	141	131	52	62	63	49
24	27	26	21	139	133	136	138	53	59	58	56
28	23	22	25	135	137	140	134	57	55	54	60
17	30	31	20	142	132	129	143	64	50	51	61

In like manner [redacted] 15^2 equal to 9 multiplied by 25, 16^2 equal to 16 multiplied by 16, 18^2 equal to 9 multiplied by 36, etc., containing as many squares as terms in one of the factors, can be written at once by combination of the small models.

SQUARES WHOSE LAST TERM IS DIVISIBLE BY 4.

If the central 16 terms of the series of natural numbers from 1 to 36 be developed into a magic square, then the remaining terms can be so combined as to construct a magic square of the series.

For let this series be arranged one-half over the other so that the sum of each pair is 37, the sum of the extremes.

								DIAGONALS.		CENTRAL SERIES.							
1	2	3	4	5	6	7	3	9	10	11	12	13	14	15	16	17	18
36	35	34	33	32	31	30	29	28	27	26	25	24	23	22	21	20	19

If we construct the square of the central terms from 11 to 26, the constant number is 74, to which if 37 be added we obtain 111, the constant number of the series. To complete the square, take the terms adjacent to those already placed for the diagonals and combine the remaining terms in pairs so as to make the sum of the terms of each side of the exterior band 111, thus constructing the square.

9	29	30	31	2	10
36	11	24	25	14	1
3	18	21	20	15	34
4	22	17	16	19	33
32	23	12	13	26	5
27	8	7	6	35	28

The constant number of the interior square is twice the sum of one pair of terms, and that of the series is three times the same sum; therefore, they are to each other as 2 to 3.

If, in the square of 8, we construct the interior square of the central group of terms from 25 to 40, then the constant numbers of the respective squares are two, three, and four times the sum of the extremes, or 130, 195 and 260, and in like manner to the square of six the exterior bands can be formed of pairs of terms, the sum of each pair being 65.

SQUARE OF 12.

124	130	14	132	12	134	10	136	8	138	5	27
22	103	104	107	114	116	30	32	37	39	43	123
20	40	92	94	96	46	100	48	50	54	105	125
126	33	93	82	62 (d)	61	60	89 (c)	81	52	112	19
17	28	98	88	65	78	79	68	57	47	117	128
16	127	101	87	72	75	74	69	58	44	18	129
139	26	36	59	76	71	70	73	86	109	119	6
141	120	35	55	77	66	67	80	90	110	25	4
3	24	34	64	83 (a)	84	85	56 (b)	63	111	121	142
143	122	91	51	49	99	45	97	95	53	23	2
1	102	41	38	31	29	115	113	108	106	42	144
118	15	131	13	133	11	135	9	137	7	140	21

METHOD OF CONSTRUCTION.

It is impossible to make a magic square of 4 terms, therefore the square of 4 is the least interior square, the number of magic squares is six, and in the greatest square there are six ***pairs*** of rows or columns.

We find the first term, 65, of the central magic square by subtracting 7 from half the greater extreme, and developing the series we have the square, a, b, c, d, from which the following results are obtained:

First. The constant number is 290, or 145 for each pair of rows or columns.

Second. The sum of two terms along each diagonal equidistant from the center is 145, and they are alternately odd and even numbers.

Third. The differences between the terms of the diagonals of the opposite sides of the square of 4 are equal.

If these laws are applied to the outer squares, then their constant number is increased by the successive addition of 145, and the result, 870, is the constant number of the square of 12, and the sum of two diagonal terms of each band is 145. These facts are verified by calculation; they are invariable, and are the properties of numbers by which each remaining square is completed.

The constant number of the second square, deduced from the foregoing analysis, is 290 plus 145 or 435. No rule is observed in the choice of numbers for the diagonals; make their sum in pairs 145, then make the sum of each outside row and column 435, selecting pairs for the centre of rows and columns whose sum is 145.

In like manner complete each band in successive order, taking especial care to avoid errors in addition or duplication of terms.

Since the constant numbers of the square are found by successive additions of twice the half sum of the extremes 1 and 144, therefore, they are to each other as the even numbers, 4, 6, 8, 10 and 12.

By this construction, if the exterior bands be successively removed, the remaining square is magical.

If in the central square of 6, Fig. 1, we develop the middle series from 9 to 17, or in Fig. 2, the series odd terms from 5 to 21, then in each the constant number is 3 times the half sum of the extremes of the series of the square of 5, in which (8) the sum of any two terms equidistant from the centre is twice the same half sum.

FIG. 1.

5	18	20	3	19
24	14	9	16	2
25	15	13	11	1
4	10	17	12	22
7	8	6	23	11

NO. 2.

18	10	3	22	12
6	15	5	19	20
25	17	13	8	1
2	7	11	11	24
14	16	23	4	8

Complete the magic square by the same law of terms, filling the cells so as to make the sum of each side equal to 65.

In like manner, in the central square of 7, Fig. 3, develop the middle series of odd terms from 17 to 33, or in Fig. 4, the series from 21 to 29, and the resultant constant number is 75. The bands are filled as described in the square by adding twice 25 to each in succession so as to obtain respectively the constant numbers 125 and 175.

FIG. 3.

26	48	20	1	40	8	32
34	28	35	3	13	46	16
38	45	27	17	31	5	12
39	7	29	25	21	43	11
14	41	19	33	23	9	36
6	4	15	47	37	22	44
18	2	30	49	10	42	24

FIG. 4.

13	4	36	41	40	3	38
42	18	39	20	15	33	8
6	43	26	21	28	7	44
5	31	27	25	23	19	45
48	16	22	29	24	34	2
49	17	11	30	35	32	1
12	46	14	9	10	47	37

Fig. 5 is the square of 13 constructed in the same manner, the central square containing the middle series of odd terms from 77 to 93 and the even terms removed to the corners of the square.

FIG. 5.

84	152	80	134	114	78	1	158	58	142	24	166	26
168	88	82	74	64	145	163	19	50	138	34	98	2
154	102	100	20	15	133	29	17	149	162	140	68	16
22	18	164	109	119	35	43	53	115	121	6	152	148
54	42	161	45	101	75	65	73	111	125	9	128	116
130	31	5	123	99	87	77	91	71	47	165	139	40
39	151	11	113	63	89	85	81	107	57	159	13	131
14	137	167	129	103	79	93	83	67	41	3	33	156
94	46	23	27	59	95	105	97	69	143	147	124	76
10	110	104	49	51	135	127	117	55	61	66	60	160
48	132	30	150	155	37	141	153	21	8	70	38	122
44	72	108	96	106	25	7	151	120	32	136	82	126
144	118	90	36	56	92	169	12	112	28	146	4	86

From the foregoing examples of squares whose roots are odd we deduce the following rules and laws of numbers:

FIRST. The term in the central cell is the central odd number of the series, from which if 4 be deducted the remainder is the first term of the series of 9 terms in the central square.

SECOND. If the number in the central cell be multiplied by the odd numbers 3, 5, 7, 9, etc., the several products will be the constant numbers of the respective squares from the centre outward.

THIRD. The required summation of the bands is made by selecting pairs of terms whose sum is double the number in the central cell. The terms of the pairs are placed on opposite sides of the square.

FOURTH. If one or more exterior bands of numbers are removed the remaining interior squares are magical.

The work proposed has been done. Magic squares can be made ad infinitum.

ASTRONOMICAL PROBLEM.

The following rule for finding the true meridian by observations on the North Star at any time of night was deduced by me from Spherical Trigonometry applied to astronomy, while employed in the public surveys of Montana.

Since it is not the design to teach, but to give a brief, practical rule for practical men, demonstration is omitted.

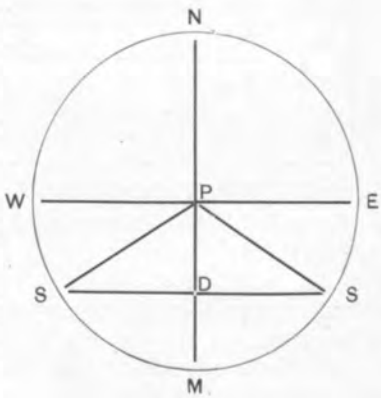
Two astronomical facts must be known.

The TIME when the star is on the meridian on the day of observation, and the ANGLE subtended at the place of observation by the greatest distance or elongation of the star from the pole.

For this information refer to Davies' Surveying or other surveyor's manuals.

RULE.—Convert the observed time computed after or before meridional passage into angular movement of the star at the rate of 4 minutes to a degree, and multiply the natural sine of this angle by the elongation.

EXAMPLE. — Let M E N W, be the circle described by the star Polaris around the pole P, M P N the meridian of the observer, P E the eastern elongation 53 minutes, and S the place of the star, 3 hours 37 minutes after meridional passage. Converting time to angular motion, D P S is equal to 54 degrees 15 minutes.



Applying natural sines..... .81157
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D S is equal to ..... 43.01321 minutes.

The course of a line from the observer to the pole star is therefore, N. 0 degrees 43 minutes E.

B. F. MARSH.

ANALYSIS

OF THE

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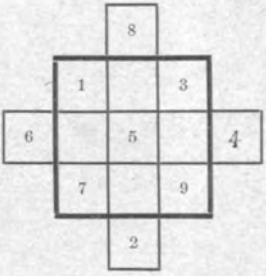
CONSTRUCTION OF

MAGIC SQUARES.



5
Arithmetic

BY

BENJAMIN F. MARSH, C. E.

TOGETHER WITH METHOD OF FINDING TRUE MERIDIAN BY OBSERVATION ON POLARIS.

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